**NUMERICAL METHODS   
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 4. Minimization of functionals

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The easiest extremum problem is the problem of minimization for the function of one variable. It can be analyzed with using of the stationary condition and gradient method. These results use the differentiation of the given function. We know the method the differentiation of the general functionals. So we can try to extend the known methods of the functions minimization to minimization problems of the functionals.

### 4.1. Stationary conditions for functionals

Let us consider a general functional *I* on the unitary space *V*. We have the problem of its minimization.

**Theorem** **4.1**. *Let u be the point* *of a minimum of Gateaux differentiable functional I on the space with scalar product V. Then it satisfies the* ***stationary condition***

 (4.1)

**Proof**. If *u* is the point of a minimum of the functional *I* on the set *V*, then we have the inequality



So we get

 (4.2)

Choose a positive number *σ.* After division by *σ* and passing to the limit as  with using of the differentiability of the functional *I* we obtain



However we can choose a negative number *σ* at the inequality (4.2). So after division by *σ* and passing to the limit we have



Using to last inequalities we get



This formula is true for all elements *h* of the space *V*. We can choose particularly  Then we obtain



So the equality (4.1) is true because of the property of the norm.

**Definition 4.1**. *The equality* (4.1) *is called stationary condition.*

Consider examples.

**Example 4.1**. *The function of one variable.* Consider a function  We know (see Example 3.1) that its Gateaux derivative at the point *x* is its classical derivative at this point. Then stationary condition (4.1) at the point *x* has the form  This is the classical stationary condition for the function of one variable.

**Example 4.2**. *The function of many variables.* Consider a function  We know that Gateaux derivative of the function *f* of many variable at the point  is its gradient



at this point. Then we determine the stationary condition (4.1)



It can be transform to



So the stationary condition for a function of many variables is the system of nonlinear algebraic equations.

**Example 4.3**. *Lagrange functional.* Consider the functional



on the set *V* of the smooth enough functions on the interval  with zero values on the boundary of this interval, where *F* is a smooth enough function. Gateaux derivative of this functional at a point (function) *u* is



So we have the stationary condition (4.1)



This equality is true for all point *x* from given interval. This is second order ordinary differential equation. It is called Euler equation. So the stationary condition for Lagrange integral is Euler equation. This second order differential equation can be solved with two homogenious boundary conditions because we consider the functions with zero values on the boundary of the given interval.

**Example 4.4.** *Dirichlet integral*. LetΩbe *n-*dimensional set with the boundary *S.* Consider the integral



where *f* is a given function. Gateaux derivative of this functional at the point *u* is



So we have the stationary condition (4.1)



This equality is true for all point *x* of the set Ω. This partial differential equation is called Poisson equation. We consider the functions with zero value on the boundary of the given set. So we have homogeneous first order boundary condition. The corresponding boundary problem is called the homogeneous Dirichlet problem. Hence the stationary condition for the Dirichlet integral is Homogeneous Dirichlet problem for Poisson equation.

Table 1.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **argument** | number | vector | function  of one variable | function  of many variables |
| **functional** |  |  | Lagrange | Dirichlet |
| **space** | set of real |  |  |  |
| **derivative** |  |  |  |  |
| **zero** |  |  |  |  |
| **equation** |  |  |  |  |

### 4.2. Gradient methods for problems of functionals minimization

Выход на градиентный метод через нелинейные алгебраические уравнения. Картинка ждя функции одной переменной.

Let us have the problem of minimization of the differentiable functional *I* on the unitary space *V*. We can use gradient method

 (4.3)

where the iterative parameter  is positive. We can return to the considered before functionals.

**Example 4.1**. *The function of one variable.* Consider the problem of the minimization of a function  The equality (4.3) can be transformed to

 (4.4)

This is the known formula of gradient method for the function of one variable.

**Example 4.2**. *The function of many variables.* Consider the problem of the minimization of a function  We know that its Gateaux derivative at the point is the gradient



So we have the vector form of the equality (4.4), that is



**Example 4.3**. *Lagrange functional.* Consider the problem of the minimization of the functional



on the set *V* of the smooth enough functions on the interval  with zero values on the boundary of this interval. We know its Gateaux derivative



So we have the gradient method



**Example 4.4.** *Dirichlet integral*. LetΩbe *n-*dimensional set with the boundary *S.* Consider the problem of the minimization of the integral



Gateaux derivative of this functional at the point *u* is



Then we have the gradient method



We can ignore here the constant 2 before iterative parameter because  can be denoted be the new iterative parameter.

**Example 4.5.** *Square of the norm for the unitary space.* We have the problem of the minimization of the functional



Its Gateaux derivative at the point u is

1. 

So we get the gradient method



We ignore here the constant 2.

### Next step

We know that inverse problems can be transformed to the minimization problems. The easiest minimization problem is the problem of the minimization of the function of one variable. It can be solve with using of the stationary condition and gradient method. This technique is based on the differentiation of the given function. Now are able to differentiate general functionals. So we can use the known optimization methods to problems of minimization general functionals. However the minimizing functional can depends from the sought for parameter indirectly. For the standard inverse problem the functional depends from the state function, which depends from the known parameter by a state equation. We will try to extend our optimization methods to this case.

**Task**

Determine stationary condition and iterative method from the extremum problem of the previous task

Добавить: выпуклость и достаточность